Commutative Algebra
Mid Terminal Examination

September 12 2018

This exam is of **30 marks**. Please **read all the questions carefully** and **do not cheat**. Please feel free to use whatever theorems you have learned in class after stating them clearly. Please **hand in your phones** at the beginning of the class.

1. Let A be a ring and $A[X]$ the polynomial ring. Let $f \in A[X]$. Show (6)

$$f^n = 0 \text{ for some } n \in \mathbb{N} \Leftrightarrow 1 - fg \text{ is a unit for all } g \in A[X]$$

2. Let A be a ring. Show that $A[X]$ is **flat** as an A-module. (4)

3. If A is a ring, show that A is naturally an $A[X]$-module. (2)

4. Is $\mathbb{Z}$ a **flat** $\mathbb{Z}[X]$-module? (4)

5. Does there exist a ring with exactly 3 maximal ideals. If so, construct an example. (4)

6. Does there exist a ring A with exactly 1 maximal ideal, but infinitely many prime ideals? If so, construct an example. (4)

7. Let A be a ring and $\mathfrak{p}$ a prime ideal. Let $A_{\mathfrak{p}} = S^{-1}A$, where $S = A - \mathfrak{p}$. Let

$$S_{\mathfrak{p}}(0) = \ker(A \longrightarrow A_{\mathfrak{p}})$$

Prove

- $S_{\mathfrak{p}}(0) \subset \mathfrak{p}$ (2)
- $r(S_{\mathfrak{p}}(0)) = \mathfrak{p} \Leftrightarrow \mathfrak{p}$ is a minimal prime ideal of A. (4)

Extra Credit: Consider the ring $C([0,1], \mathbb{R})$ of continuous real valued functions. Is the ideal (0) decomposable in this ring? Prove your answer. (10)